Impact of fear and Beddington-DeAngelis functional response in a Predator-Prey Model

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Abstract

This paper studies a predator-prey model incorporating prey refuge and fear effect. The functional response is considered to be of Beddington-DeAngelis type. The dynamics of the system is discussed mainly from the point of view of permanence and stability. We obtain conditions that affect the persistence of the system. Local and global asymptotic stability of various equilibrium solutions is explored to understand the dynamics of the model system. The global asymptotic stability of positive interior equilibrium solution is established using suitable Lyapunov functional. We investigate the role of fear in the dynamics of system, it is found that Hopf bifurcation occurs when the fear parameter k crosses some critical value. Furthermore, the deterministic model has been extended to a stochastic model by introducing environmental white noise and jump process. It has been observed that the stochastic system possesses a unique globally stable positive solution. The stochastic extinction and persistence scenario for both the species have been analysed. Finally, numerical examples are introduced to check the theoretical results.

Key words: Fear effect; Prey refuge; Hopf-bifurcation; Lévy noise

1 Introduction

As an essential component of ecology, population models have been widely studied and explored for their rich dynamic properties, with the aim of providing theoretical guidance for the conservation, exploitation and utilization of biological resources [1]. Out of the most important population models, predator-prey models play an important role in understanding the interactions between different species in unstable natural environments. For ecologists, biologists and mathematicians, modeling and analysis of such systems is a very interesting and active research topic. Over the years, many people have made great contributions to this subject [2–4].

In the ecosystem, predator and prey interaction is one of the most fundamental factors in shaping community structure and maintaining ecological diversity. To capture the effects of predators on prey populations, two different approaches exist. One is the consumption of prey (direct effect) by predators [5], which is easier to observe in the field and has been the main focus of mathematical ecology so far. Another factor is the fear of predator on prey animals (indirect effect), which may alter prey demography. A growing body of evidence suggests that fear of predators has a greater impact than direct consumption and that it plays a crucial role in the dynamics of predator-prey interactions. When the prey population perceives the predator signal (chemical/vocal), they often spend more time being vigilant and less time in foraging [6]. They also shift to a safer place from higher predation risk areas for foraging with perceived to a lower predation risk [7,8] and sacrifice their higher grazing zone. Such behavioral changes are due to fear, which can lead to physiological stress on prey species and have a negative influence on their reproduction strategies and long-time survival. For example, in 2011, the experiment of Zanette et al. [9] showed that the song sparrows (Melospiza Melodia) produced 40% fewer offspring due to fear of predators. This reduction is due to the effect of anti-predator behavior on birth rates and offspring survival. Hua et al. [10] manipulated vocalized predatory traits and

observed that breeding bluebirds can regulate their reproductive strategies and actions accordingly. Laundre et al. [11] showed that elk increases vigilance and their feeding rate decreases due to the reintroduction of wolves into Yellowstone. Suraci et al. [12] have induced fear among mesocarnivores by using month-long playbacks of large carnivores sound and showed that fear of large carnivore reduced mesocarnivore foraging behavior (lower by 66%) and increased vigilance. Thus, theoretical biologists and evolutionary ecologists have realized that not only direct kills or shocks need to be considered, but indirect shocks or fear costs must be factored into predator and prey population models.

On the other hand, in reality, not all prey are captured by predators because they usually have refuges to avoid predators [13, 14]. In order to effectively avoid predators and increase the survival rate, the prey species often actively seek refuge. This phenomenon, known as the refuge effect [15,16], has been scientifically demonstrated for a long time. Mukherjee, D. [17] showed that prey sanctuary thresholds in food chain models can determine long-term survival for all species; furthermore, over-execution of predator defense strategies may be the root cause of predator extinction from the system. The prey's sanctuary depends on prey biomass and the number of predators. Predator systems with prey refuges are also one of the hotspots in biomathematical research, and many scholars have made great achievements in this field [18–21].

The behavioral characteristics of predation can be called functional responses, which play a dominant role in some complex dynamical behaviors, such as the steady states, bistability, periodic oscillations, chaos and bifurcation phenomena. The functional response depends on many factors, such as different prey densities, the efficiency with which predators search for and kill prey, processing time, competition between predators, etc. Traditional prey-dependent functional responses fail to mimic predator-predator interference and face challenges from biological and physiological communities. Some biologists have argued that in many cases, especially when predators must search for food (and therefore must share or compete for food), the functional response in the predator-prey model should be predator-dependent, and numerous experiments and observations have shown this to be the case. In order to reconcile the theoretical and experimental views, Beddington and DeAngelis et al. considered a functional form of prey consumption rate and proposed the following form, $f(x, y) = \frac{px}{ax+by+c}$, which is similar to Holling type II functional response, but there is an extra term "by" in the denominator, which is interpreted as an interference between predators. The function $f(x, y) = \frac{px}{ax+by+c}$ is called the Beddington-DeAngelis function response [22–24].

Motivated by these facts, we first propose a deterministic model with Beddington-DeAngelis functional response along with the fear factor of prey induced by predators and prey refuge. Then we extend our deterministic model to the stochastic model by incorporating white noise terms and Lévy noise terms. The paper is organised as follows. In the next section, we consider the existence conditions of the positive equilibrium, persistence and global stability of coexistence equilibrium of the deterministic model system. In Section 3, we extend the deterministic model to the stochastic model analyzing the existence, uniqueness and boundedness of global positive solutions and discussing the conditions for stochastic extinction and persistence of both species. In Section 4, we also perform some carefully designed numerical simulations to validate our analytical findings. We close the paper with a conclusion in Section 5.

2   Deterministic model

We consider an ecological system consisting of a single prey and single predator species. Let $x(t)$ be the prey density at time t, and $y(t)$ be the predator population density at time t. It is assumed that the predator preys on prey according to the functional response of Beddington-DeAngelis. In the presence of direct predation and fear factor, the prey population follows a logical growth, then we get the following differential system:

$$\frac{dx}{dt} = x\left(\frac{\alpha}{1+ky} - bx - \frac{c_1(1-m)y}{1+a_1(1-m)x+a_2y}\right) = xf_1(x, y), \qquad (2.1)$$

$$\frac{dy}{dt} = y\left(-d - ey + \frac{c_2(1-m)x}{1+a_1(1-m)x+a_2y}\right) = yf_2(x, y), \qquad (2.2)$$

with $x(0) = x_0 > 0, y(0) = y_0 > 0$, where α, b, c_1, c_2, d, e, a_1, a_2, k, m are positive constants, α is the intrinsic growth rate of prey, d is the natural death rate of the predator, b and e respectively represent mortality rates of the prey and predator species due to intraspecific competition between individuals. The

constants a_1 and a_2 are the half-saturation constant of prey and predator, respectively. c_1 is the rate of predation and c_2 is the conversion rate of prey to predator [25, 26]. Function $g(k, y) = \frac{1}{1+ky}$ represents the fear function which stands for the cost of anti-predator defence of prey due to fear induced by predator and k is the level of fear. The function $g(k, y)$ has some special properties, as several field data show that the effect of fear reduces the reproductive process of prey species. For more details on the fear function $g(k, y)$, see [27]. m is the strength of prey refuge and $m \in [0, 1)$, thus $(1 - m)x$ is only prey available to predator.

2.1 Positivity and boundedness of the solutions

Let $\mathbb{R}_+$ denote the set of all non-negative real numbers and $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x = (x_1, \ldots, x_n)$ where $x_i \in \mathbb{R}_+, \forall \ i = 1, 2, \ldots, n\}$. If we denote the function on the right hand of System (2.1) and (2.2), by $\mathbf{F} = (F_1, F_2)$, clearly, $\mathbf{F} \in C^1(\mathbb{R}_+^2)$. Hence, $\mathbf{F} : \mathbb{R}_+^2 \to \mathbb{R}^2$ is locally Lipschitz on $\mathbb{R}_+^2 = \{(x, y) : x \geqslant 0, y \geqslant 0\}$. Thus the fundamental theorem of existence and uniqueness assures existence and uniqueness of solution of System (2.1) and (2.2) with the given initial condition. The state space of the system is the non-negative cone, $\mathbb{R}_+^2 = \{(x, y) : x \geqslant 0, y \geqslant 0\}$. In the theoretical ecology, positivity and boundedness of the system establishes the biological well behaved nature of the system. The following results ensure the positivity, boundedness, dissipativeness and permanence of solutions of deterministic System (2.1) and (2.2) [28].

Lemma 2.1 *The positive quadrant* $(Int(\mathbb{R}_+^2))$ *is invariant for System (2.1) and (2.2).*

Proof. Here, we wish to prove that for all $t \in [0, P]$, $x(t) > 0$, $y(t) > 0$, where P is any positive real number. We show this by method of contradiction. Suppose this is not true. Hence, there must exists one t_p, $0 < t_p < P$. s.t. $\forall \ t \in [0, t_p]$, $x(t) > 0$, $y(t) > 0$ and at least one of $x(t_p)$, $y(t_p)$ must vanish. From System (2.1) and (2.2), we have

$$x(t) = x(0) \exp\left(\int_0^t (f_1(x(s), y(s)))ds\right),$$

$$y(t) = y(0) \exp\left(\int_0^t (f_2(x(s), y(s)))ds\right).$$

Since (x, y) are defined and continuous on $[0, t_p]$, there exist a $Q \geqslant 0$ such that $\forall \ t$ in $[0, t_p]$,

$$x(t) = x(0) \exp\left(\int_0^t (f_1(x(s), y(s)))ds\right) \geqslant x(0) \exp(-t_p Q),$$

$$y(t) = y(0) \exp\left(\int_0^t (f_2(x(s), y(s)))ds\right) \geqslant y(0) \exp(-t_p Q).$$

It is clear that if $t \to t_p$ we obtain

$$x(t_p) \geqslant x(0) \exp(-t_p Q),$$

$$y(t_p) \geqslant y(0) \exp(-t_p Q),$$

which contradicts the fact that at least one of $x(t_p)$ and $y(t_p)$ must vanish. So, $\forall \ t \in [0, P]$, $x(t) > 0$, and $y(t) > 0$. $\qquad \square$

Now we discuss about the conservation of overall energy or biomass flow i.e. we search for whether there is some region in the dynamical space within which System (2.1) and (2.2) may be bounded. To prove uniform boundedness of System (2.1) and (2.2), we use the following comparison lemma [29, 30].

Lemma 2.2 (Comparison lemma). *Let $s(t)$ be an absolutely continuous function which satisfies the differential inequality:*

$$\frac{d(s(t))}{dt} + u_1 s(t) \leqslant u_2, \ \forall \ t \geqslant 0,$$

where $(u_1, u_2) \in \mathbb{R}^2$ and $u_1 \neq 0$. Then there exist $\hat{T} \geqslant 0$ such that for all $t \geqslant \hat{T}$, $s(t) \leqslant \frac{u_2}{u_1} - \left(\frac{u_2}{u_1} - s(\hat{T})\right)e^{-u_1(t-\hat{T})}.$

Theorem 2.1 *Let $\eta(t) = x(t) + y(t)$. Then all the solutions of System (2.1) and (2.2) starting in $\mathbb{R}_+^2$ are confined to the region $D^* = \{(x, y) \in \mathbb{R}_+^2 : x(t) \leqslant 1, 0 \leqslant \eta(t) \leqslant \frac{(\alpha+1)^2}{4b} + \frac{(1-d)^2}{4e}\}$ as $t \to \infty$ for all positive initive value $(x(0), y(0)) \in \mathbb{R}_+^2$.*

Proof. Differentiating η with respect to time t along the solution of System (2.1) and (2.2), we obtain

$$\frac{d\eta(t)}{dt} = \frac{dx(t)}{dt} + \frac{dy(t)}{dt},$$

$$= x(t)\left(\frac{\alpha}{1+ky(t)} - bx(t)\right) - y(t)(d + ey(t)),$$

$$\leqslant x(t)(\alpha - bx(t)) - y(t)(d + ey(t)),$$

$$= -(x(t) + y(t)) + bx(t)\left(\frac{\alpha+1}{b} - x(t)\right) + ey(t)\left(\frac{1-d}{e} - y(t)\right),$$

$$\leqslant -\eta(t) + \left(\frac{(\alpha+1)^2}{4b} + \frac{(1-d)^2}{4e}\right).$$

Thus, we have

$$\frac{d\eta(t)}{dt} + \eta(t) \leqslant \frac{(\alpha+1)^2}{4b} + \frac{(1-d)^2}{4e}.$$

Using comparison Lemma 2.2, we have $\eta(t) \leqslant \frac{(\alpha+1)^2}{4b} + \frac{(1-d)^2}{4e}$. Thus all the solutions of System (2.1) and (2.2) are uniformly bounded with an ultimate bound. $\square$

2.2 Dissipativeness and permanence

In this subsection, we analyze the dissipativeness, persistence (weak and strong) and permanence behaviour of System (2.1) and (2.2). Persistence and permanence are important behaviour of the system in the sense that they describe long term behaviour of the system. Analytically, a system is said to be persistent if it persists for each of the populations i.e. $\liminf_{t\to\infty} z(t) > 0$ (stronger case) or $\limsup_{t\to\infty} z(t) > 0$ (weaker case) for each of the populations $z(t) = x(t)$ or $y(t)$ of the system. Geometrically, persistence means that trajectories that initiate in a positive cone are eventually bounded away from co-ordinate axes. On the other hand, permanently coexistence (uniform persistence) implies the existence of a region in the phase space at a non-zero distance from boundary in which all the population vectors must lie ultimately. The latter case assures the survival of species in biological sense.

Definition 2.1 (Persistence). System (2.1) and (2.2) is said to be weakly persistent if every solution $(x(t), y(t))$ satisfies two conditions:

  i. $x(t) \geqslant 0$, $y(t) \geqslant 0$, $\forall\, t \geqslant 0$.

  ii. $\limsup_{t\to\infty} x(t) > 0$, $\limsup_{t\to\infty} y(t) > 0$.

System (2.1) and (2.2) is said to be strongly persistent if every solution $(x(t), y(t))$ satisfies the following condition along with the first condition of the weak persistence:

$$\liminf_{t\to\infty} x(t) > 0, \quad \liminf_{t\to\infty} y(t) > 0.$$

Definition 2.2 (Permanence and non-permanence). System (2.1) and (2.2) is said to be permanent if $\exists$ positive constants M_1, and M_2, with $0 < M_1 \leqslant M_2$ such that

$$\min\left\{\liminf_{t\to\infty} x(t),\; \liminf_{t\to\infty} y(t)\right\} \geqslant M_1, \quad \max\left\{\limsup_{t\to\infty} x(t),\; \limsup_{t\to\infty} y(t)\right\} \leqslant M_2,$$

for all solutions $(x(t), y(t))$ of System (2.1) and (2.2) with positive initial values. Model system (2.1) and (2.2) is said to be non-permanent if there is a positive solution $(x(t), y(t))$ of (2.1) and (2.2) s.t.

$$\min\left\{\limsup_{t\to\infty} x(t),\; \limsup_{t\to\infty} y(t)\right\} = 0.$$

To establish the persistence for System (2.1) and (2.2), we need to recall the following lemma, whose proof can be found in [31].

Lemma 2.3 *If $p > 0$, $q > 0$ and $\frac{du}{dt} \leqslant (\geqslant)u(t)(q - pu(t))$, $u(t_0) > 0$, then $\limsup_{t\to\infty} u(t) \leqslant \frac{q}{p} \left(\liminf_{t\to\infty} u(t) \geqslant \frac{q}{p} \right)$.*

Since the dependent variables are positive, from (2.1), it is easy to see

$$\frac{dx}{dt} \leqslant x(\alpha - bx),$$

Using Lemma 2.3, we get

$$\limsup_{t\to\infty} x(t) \leqslant \frac{\alpha}{b} \equiv K.$$

Thus for arbitrary $\epsilon_1 > 0$, thers exist a positive real number T_1 such that

$$x(t) \leqslant K + \epsilon_1, \ \forall \ t \geqslant T_1.$$

Further, form (2.2), we have

$$\frac{dy}{dt} \leqslant y \left(-d - ey + \frac{c_2}{a_1} \right),$$

which gives

$$\limsup_{t\to\infty} y(t) \leqslant \frac{c_2 - a_1 d}{a_1 e} \equiv L,$$

by using Lemma 2.3 provided $c_2 > a_1 d$. Hence, for arbitrary positive real number $\epsilon_2 > 0$, $\exists$ a positive real number T_2 such that

$$y(t) \leqslant L + \epsilon_2, \ \forall \ t \geqslant T_2.$$

Then from (2.1), we have

$$\frac{dx}{dt} \geqslant x \left(\frac{\alpha}{1 + k(L + \epsilon_2)} - bx - \frac{c_1(1 - m)}{a_2} \right).$$

Using Lemma 2.3, we obtain

$$\liminf_{t\to\infty} x(t) \geqslant \left(\frac{\alpha}{b(1 + kL)} - \frac{c_1(1 - m)}{ba_2} \right) \equiv M, \ \text{provided} \ a_2 > \frac{c_1(1 - m)}{\alpha} \left(1 + \frac{k(c_2 - a_1 d)}{a_1 e} \right).$$

For arbitrary $\epsilon_3 > 0$, $\exists$ a positive real number T_3 such that

$$x(t) \geqslant M - \epsilon_3, \ \forall \ t \geqslant T_3.$$

Moreover, using the lower and upper bounds for x and y, from (2.2), we obtain

$$\frac{dy}{dt} \geqslant y \left(-d - ey + \frac{(1 - m)c_2(M - \epsilon_3)}{1 + a_1(1 - m)(M - \epsilon_3) + a_2(L + \epsilon_2)} \right),$$

for sufficiently large $t \geqslant T$, where $T = \max(T_1, T_2, T_3)$. Thus, using Lemma 2.3 and the arbitrariness of $\epsilon_i > 0$, $\forall \ i = 1, 2, 3$, we obtain

$$\liminf_{t\to\infty} y(t) \geqslant \frac{(1 - m)c_2 M}{e(1 + a_1(1 - m)M + a_2 L)} - \frac{d}{e} \equiv N,$$

provided $(c_2 - a_1 d)\left[\frac{1-m}{b} \left(\alpha \left(1 + \frac{k(c_2 - a_1 d)}{a_1 e} \right)^{-1} - \frac{c_1(1-m)}{a_2} \right) - \frac{a_2 d}{a_1 e} \right] > d.$

The above results can be summarised into the following theorems.

Theorem 2.2 *If $c_2 > a_1 d$, then any solution of (2.1) and (2.2) starting from the interior of the first quadrant satisfies the following inequalities:*

$$\limsup_{t\to\infty} x(t) \leqslant \frac{\alpha}{b} \equiv K,$$

$$\limsup_{t\to\infty} y(t) \leqslant \frac{c_2 - a_1 d}{a_1 e} \equiv L.$$

Theorem 2.3 *If $a_2 > \frac{c_1(1-m)}{\alpha}\left(1 + \frac{k(c_2-a_1 d)}{a_1 e}\right)$ and $(c_2 - a_1 d)\left[\frac{1-m}{b}\left(\alpha\left(1 + \frac{k(c_2-a_1 d)}{a_1 e}\right)^{-1} - \frac{c_1(1-m)}{a_2}\right) - \frac{a_2 d}{a_1 e}\right] > d$ are satisfied, then any solution of (2.1) and (2.2) starting from the interior of the first quadrant satisfies the following inequalities:*

$$\liminf_{t\to\infty} x(t) \geqslant \left(\frac{\alpha}{b(1+kL)} - \frac{c_1(1-m)}{ba_2}\right) \equiv M,$$

$$\liminf_{t\to\infty} y(t) \geqslant \frac{(1-m)c_2 M}{e(1 + a_1(1-m)M + a_2 L)} - \frac{d}{e} \equiv N.$$

Remark 2.1 Theorem 2.3 along with Definition 2.1 ensures that System (2.1) and (2.2) is strongly persistent provided the conditions of Theorem 2.3 hold. Since conditions of Theorem 2.3 also ensure that $\limsup_{t\to\infty} x(t) > 0$ and $\limsup_{t\to\infty} y(t) > 0$, System (2.1) and (2.2) is weakly persistent under the conditions of the Theorem 2.3.

The condition $(c_2 - a_1 d)\left[\frac{1-m}{b}\left(\alpha\left(1 + \frac{k(c_2-a_1 d)}{a_1 e}\right)^{-1} - \frac{c_1(1-m)}{a_2}\right) - \frac{a_2 d}{a_1 e}\right] > d$ implies that $a_2 > \frac{c_1(1-m)}{\alpha}\left(1 + \frac{k(c_2-a_1 d)}{a_1 e}\right)$ which ensure that $M > 0$. Therefore, one can easily observe that the condition $c_2 - a_1 d$ together with condition $(c_2 - a_1 d)\left[\frac{1-m}{b}\left(\alpha\left(1 + \frac{k(c_2-a_1 d)}{a_1 e}\right)^{-1} - \frac{c_1(1-m)}{a_2}\right) - \frac{a_2 d}{a_1 e}\right] > d$, ensure the positivity of M, N and L. Similarly, condition $\frac{a_2 d}{a_1 e} < \frac{1-m}{b}\left(\alpha\left(1 + \frac{k(c_2-a_1 d)}{a_1 e}\right)^{-1} - \frac{c_1(1-m)}{a_2}\right)$ along with condition $(c_2 - a_1 d)\left[\frac{1-m}{b}\left(\alpha\left(1 + \frac{k(c_2-a_1 d)}{a_1 e}\right)^{-1} - \frac{c_1(1-m)}{a_2}\right) - \frac{a_2 d}{a_1 e}\right] > d$ ensure that $M > 0$, $N > 0$ and $L > 0$. Thus, we arrive at the following result:

Theorem 2.4 *System (2.1) and (2.2) is permanent if it satisfies any of the following two conditions*

P_1 $c_2 - a_1 d$ and $(c_2 - a_1 d)\left[\frac{1-m}{b}\left(\alpha\left(1 + \frac{k(c_2-a_1 d)}{a_1 e}\right)^{-1} - \frac{c_1(1-m)}{a_2}\right) - \frac{a_2 d}{a_1 e}\right] > d.$

P_2 $\frac{a_2 d}{a_1 e} < \frac{1-m}{b}\left(\alpha\left(1 + \frac{k(c_2-a_1 d)}{a_1 e}\right)^{-1} - \frac{c_1(1-m)}{a_2}\right)$ and $(c_2 - a_1 d)\left[\frac{1-m}{b}\left(\alpha\left(1 + \frac{k(c_2-a_1 d)}{a_1 e}\right)^{-1} - \frac{c_1(1-m)}{a_2}\right) - \frac{a_2 d}{a_1 e}\right] > d.$

2.3   Biomass equilibria and their existence

System (2.1) and (2.2) possesses the following three equilibrium solutions

i. The trivial equilibrium $E^0 = (0, 0)$.

ii. The predator free axial equilibrium $E^1 = (\frac{\alpha}{b}, 0)$.

iii. The steady state of coexistence (interior equilibrium point) $E^* = (x^*, y^*)$.

The interior equilibrium $E^* = (x^*, y^*)$ is the point of intersection of the prey zero growth rate isocline (i.e. when $\frac{dx}{dt} = 0$) and the predator zero growth rate isocline (i.e. when $\frac{dy}{dt} = 0$) given by

$$\frac{\alpha}{1 + ky^*} - bx^* - \frac{c_1(1-m)y^*}{1 + a_1(1-m)x^* + a_2 y^*} = 0, \tag{2.3}$$

$$-d - ey^* + \frac{c_2(1-m)x^*}{1 + a_1(1-m)x^* + a_2 y^*} = 0, \tag{2.4}$$

where $x^* = \frac{a_2e(y^*)^2+(a_2d+e)y+d}{c_2(1-m)-a_1(1-m)d-a_1(1-m)ey^*}$, provided $c_2(1-m)-a_1(1-m)d-a_1(1-m)ey^* > 0$ and y^* is a positive solution of the quintic equation $A(y)^5 + B(y)^4 + C(y)^3 + D(y)^2 + Ey + F = 0$, where

$A = c_1 a_1^2(1-m)^3 e^3 k + c_2 a_2^2(1-m)be^2 k,$

$B = c_1 a_1(1-m)^3 e^2(3a_1 dk + a_1 e - 2c_2 k) + c_2 a_2^2(1-m)be^2 + c_2 a_2^2(1-m)bdek + c_2 a_2(1-m)be^2 k,$

$C = c_1(1-m)ek(c_2(1-m) - a_1(1-m)d)^2 + c_1 a_1(1-m)^2 e(a_1(1-m)de - 2dk - 2e) + \alpha c_2 a_1 a_2(1-m)^2 e^2$
$\quad + c_2 a_2^2(1-m)bde + c_2 a_2(1-m)be^2 + c_2 a_2^2(1-m)bd^2 k + c_2(1-m)be^2 k + 4c_2 a_2(1-m)bdek,$

$D = c_1(1-m)(dk+e)(c_2(1-m) - a_1(1-m)d)^2 - (1-m)e(2c_1 a_1(1-m)d + \alpha c_2 a_2)(c_2(1-m) - a_1(1-m)d)$
$\quad + \alpha c_2 a_1 a_2(1-m)^2 de + \alpha c_2 a_1(1-m)^2 e^2 + c_2 a_2^2(1-m)bd^2 + c_2(1-m)be^2 + 3c_2 a_2(1-m)bde$
$\quad + 2c_2 a_2(1-m)bd^2 k + 2c_2(1-m)bdek,$

$E = c_1(1-m)d(c_2(1-m) - a_1(1-m)d)^2 - \alpha c_2(1-m)(a_2 d + e)(c_2(1-m) - a_1(1-m)d) + \alpha c_2 a_1(1-m)^2 de$
$\quad + 2c_2 a_2(1-m)bd^2 + 2c_2(1-m)bde + c_2(1-m)bd^2 k,$

$F = \alpha c_2(1-m)d(c_2(1-m) - a_1(1-m)d) + c_2(1-m)bd^2.$

Now, $A > 0$, but we cannot say anything about the sign of B, C, D, E and F. We cannot find suitable parametric conditions regarding the existence of the positive root of the quintic equation due to complexity in the parametric expressions. By Descartes rule of sign, if $F < 0$ (i.e. $\alpha c_2(1-m)d(c_2(1-m)-a_1(1-m)d) + c_2(1-m)bd^2 < 0$), then the quintic equation possesses at least one positive root. And on this basis, the quintic equation has exactly one positive root if anyone of the following conditions holds i) $B > 0$, $C > 0$, $D > 0$, $E > 0$, ii) $B > 0$, $C > 0$, $D > 0$, $E < 0$, iii) $B > 0$, $C > 0$, $D < 0$, $E < 0$, iv) $B > 0$, $C < 0$, $D < 0$, $E < 0$, v) $B < 0$, $C < 0$, $D < 0$, $E < 0$. This establishes the uniqueness of E^*.

2.4   Dynamical behaviour: stability analysis

In this subsection, we deal with local stability, global stability and bifurcation analysis of System (2.1) and (2.2). We denote the Jacobian matrix of System (2.1) and (2.2) at the equilibrium solution E^* by J^*.

The variational matrix J^0 in a small neighbourhood of the trivial equilibrium point $E^0 = (0,0)$ is given by the dyagonal matrix

$$J^0 = \begin{pmatrix} \alpha & 0 \\ 0 & -d \end{pmatrix},$$

and so it has eigenvalues α and $-d$. Therefore, System (2.1) and (2.2) is always unstable around E^0 which is, in fact, a saddle point and whose stable manifold is y-axis while unstable subspace as well as unstable manifold is x-axis. One can notice that the interference coefficient a_2 and prey reserve m, both do not play any role in the stability of trivial equilibrium solution E^0.

Then, the Jacobian matrix J^1 in the small neighbourhood of equilibrium point $E^1 = (\frac{\alpha}{b}, 0)$ is the triangular matrix

$$J^1 = \begin{pmatrix} -\alpha & -\dfrac{k\alpha^2}{b} - \dfrac{\alpha c_1(1-m)}{b+\alpha a_1(1-m)} \\ 0 & -d + \dfrac{\alpha c_2(1-m)}{b+\alpha a_1(1-m)} \end{pmatrix}.$$

So it has a negative eigenvalue $-\alpha$ and the other eigenvalue is positive if $-d + \frac{\alpha c_2(1-m)}{b+\alpha a_1(1-m)} > 0$. Hence System (2.1) and (2.2) is always unstable around E^1 which is, in fact, a saddle point and whose stable manifold is the x-axis while unstable subspace as well as unstable manifold is the y-axis for $m < 1 - \frac{bd}{\alpha(c_2-a_1 d)}$. On the other hand, System (2.1) and (2.2) is stable around E^1 for $m > 1 - \frac{bd}{\alpha(c_2-a_1 d)}$, and the xy plane is the stable manifold for the equilibrium point E^1. Thus, one can easily notice that the prey strength of the refuge leaves positive effect on the stability of E^1.

Theorem 2.5 *The equilibrium solution $E^1 = (\frac{\alpha}{b}, 0)$ of System (2.1) and (2.2) is globally asymptotically stable if $m \geqslant 1 - \frac{bd}{\alpha c_2}$.*

Proof. Let $(x(t), y(t))$ be any positive solution of System (2.1) and (2.2). Consider the function

$$V_1(t) = \frac{1}{b}x(t) - 1 + \frac{c_1}{bc_2}y(t) - \frac{\alpha}{b^2}\ln x(t),$$

the time derivative of $V_1(t)$ along positive solution of (2.1) and (2.2) is

$$\frac{dV_1}{dt} = \frac{1}{b}\frac{\alpha x}{1+ky} - x^2 - \frac{1}{b}\frac{c_1(1-m)xy}{1+a_1(1-m)x+a_2y} - \frac{\alpha}{b^2}\frac{\alpha}{1+ky} + \frac{\alpha}{b}x + \frac{\alpha}{b^2}\frac{c_1(1-m)y}{1+a_1(1-m)x+a_2y}$$
$$- \frac{c_1e}{bc_2}y^2 - \frac{c_1d}{bc_2}y + \frac{1}{b}\frac{c_1(1-m)xy}{1+a_1(1-m)x+a_2y},$$
$$\leqslant -(x-\frac{\alpha}{b})^2 - \frac{c_1e}{bc_2}(y-0)^2 + \left(\frac{\alpha c_1(1-m)}{b^2} - \frac{c_1d}{bc_2}\right)y.$$

If $m \geqslant 1 - \frac{bd}{\alpha c_2}$, then we have $\frac{dV_1}{dt} < 0$ except at $(x(t), y(t)) = (\frac{\alpha}{b}, 0)$ and also $\frac{dV_1}{dt} = 0$ if and only if $(x(t), y(t)) = (\frac{\alpha}{b}, 0)$. Hence Lyapunov-LaSalle's invariance principle [32] implies the global asymptotic stability of E^1. □

Now we study the dynamical behaviour of System (2.1) and (2.2) around $E^* = (x^*, y^*)$. The Jacobian matrix at the positive steady state $E^* = (x^*, y^*)$ is

$$J^* = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}, \text{ where }$$

$$J_{11} = \frac{\alpha}{1+ky^*} - 2bx^* + \frac{c_1(1-m)y^*(1+a_2y^*)}{(1+a_1(1-m)x^*+a_2y^*)^2},$$

$$J_{12} = -\frac{\alpha kx^*}{(1+ky^*)^2} - \frac{c_1(1-m)x^*(1+a_1(1-m)x^*)}{(1+a_1(1-m)x^*+a_2y^*)^2} < 0,$$

$$J_{21} = \frac{c_2(1-m)y^*(1+a_2y^*)}{(1+a_1(1-m)x^*+a_2y^*)^2} > 0,$$

$$J_{22} = -\frac{c_2(1-m)x^*(1+a_1(1-m)x^*)}{(1+a_1(1-m)x^*+a_2y^*)^2} - 2ey^* - d < 0.$$

The characteristic polynomial for J^* is $\lambda^2 - tr(J^*)\lambda + \det(J^*)$ where "tr" stands for trace and "det" for determinant. Now, applying the Routh-Hurwitz criteria, the characteristic polynomial will have either negative real roots or a pair of complex conjugate root with negative real part if $tr(J^*) < 0$ and $\det(J^*) > 0$. Now $tr(J^*) = J_{11} + J_{22}$ and $\det(J^*) = J_{11}J_{22} - J_{12}J_{21}$. Hence $tr(J^*) < 0$ and $\det(J^*) > 0$ if $J_{11} < 0$ that is System (2.1) and (2.2) is stable in the small neighbourhood of the non-trivial equilibrium E^* if

$$\frac{\alpha}{1+ky^*} - 2bx^* + \frac{c_1(1-m)y^*(1+a_2y^*)}{(1+a_1(1-m)x^*+a_2y^*)^2} < 0,$$

therefore the equilibrium point $E^* = (x^*, y^*)$ is locally asymptotically stable.

Theorem 2.6 *Let* $p(x,y) = (1+a_1(1-m)x^*+a_2y^*)(1+a_1(1-m)x+a_2y)$. *If Theorem 2.4* (P_1)-(P_2) *holds and* $b > \frac{c_1a_1(1-m)^2y^*}{p(x,y)}$, $\left(\frac{k\alpha}{(1+ky)(1+ky^*)}\right)^2 < 4u\left(b - \frac{c_1a_1(1-m)^2y^*}{p(x,y)}\right)\left(e + \frac{c_2a_2(1-m)x^*}{p(x,y)}\right)$, *then the interior equilibrium solution* E^* *of System (2.1) and (2.2) is globally asymptotically stable.*

Proof. We argue by constructing suitable Lyapunov function. Consider the function $V(x,y) : \mathbb{R}^2_+ \to \mathbb{R}$, such that

$$V(x,y) = V_2(x) + uV_3(y), \tag{2.5}$$

where $V_2(x) = (x - x^* - x^*\ln\frac{x}{x^*})$, $V_3(y) = (y - y^* - y^*\ln\frac{y}{y^*})$. Here, u is a positive constant and it is defined in below. This particular type of Lyapunov function has been widely considered (see e.g. [33]). Obviously

this function is well defined and continuous on $\text{Int}(\mathbb{R}_+^2)$. $V(x, y)$ is positive in the interior of $\mathbb{R}_+^2$ except at $E^*(x^*, y^*)$ and $V(x, y)$ vanishes at $E^*(x^*, y^*)$. Further, $\frac{\partial V_2(x)}{\partial x} > 0$ when $x > x^*$, $\frac{\partial V_2(x)}{\partial x} < 0$ when $x < x^*$ and $\frac{\partial V_3(y)}{\partial y} > 0$ when $y > y^*$, $\frac{\partial V_3(y)}{\partial y} < 0$ when $y < y^*$. Hence, $V(x, y)$ takes minimum value at (x^*, y^*). Now, we evaluate the time derivative of the positive definite scalar valued functional $V(x(t), y(t))$ along the solutions of System (2.1) and (2.2). The time derivative of V_2 and V_3 along the solution of (2.1) and (2.2) after using the equilibrium equations (2.3) and (2.4) are

$$\frac{dV_2}{dt} = \frac{(x - x^*)}{x}\frac{dx}{dt} = (x - x^*)\left(\frac{\alpha}{1 + ky} - \frac{\alpha}{1 + ky^*} - b(x - x^*) + \frac{c_1(1 - m)y^*}{1 + a_1(1 - m)x^* + a_2y^*}\right.$$
$$\left. - \frac{c_1(1 - m)y}{1 + a_1(1 - m)x + a_2y}\right),$$

$$\frac{dV_3}{dt} = u\frac{(y - y^*)}{y}\frac{dy}{dt} = u(y - y^*)\left(-e(y - y^*) - \frac{c_2(1 - m)x^*}{1 + a_1(1 - m)x^* + a_2y^*} + \frac{c_2(1 - m)x}{1 + a_1(1 - m)x + a_2y}\right).$$

$p(x, y) = (1 + a_1(1 - m)x^* + a_2y^*)(1 + a_1(1 - m)x + a_2y)$, differentiating (2.5) and replacing the values of $\frac{dV_2}{dt}$ and $\frac{dV_3}{dt}$, and after some algebraic manipulations, we obtain

$$\frac{dV}{dt} = (x - x^*)\left\{-\frac{k\alpha(y - y^*)}{(1 + ky)(1 + ky^*)} - b(x - x^*) + \frac{c_1(1 - m)(x - x^*)\left[-(y - y^*) + a_1(1 - m)(xy^* - x^*y)\right]}{p(x, y)}\right\}$$
$$+ u(y - y^*)\left\{-e(y - y^*) + \frac{uc_2(1 - m)(y - y^*)\left[(x - x^*) + a_2(xy^* - x^*y)\right]}{p(x, y)}\right\},$$

$$= (x - x^*)\left\{-\frac{k\alpha(y - y^*)}{(1 + ky)(1 + ky^*)} - b(x - x^*)\right.$$
$$+ \left.\frac{c_1(1 - m)(x - x^*)\left[-(y - y^*) + a_1(1 - m)(y^*(x - x^*) - x^*(y - y^*))\right]}{p(x, y)}\right\}$$
$$+ u(y - y^*)\left\{-e(y - y^*) + \frac{uc_2(1 - m)(y - y^*)\left[(x - x^*) + a_2(y^*(x - x^*) - x^*(y - y^*))\right]}{p(x, y)}\right\},$$

$$= -b(x - x^*)^2 - eu(y - y^*)^2 + \frac{c_1a_1(1 - m)^2y^*}{p(x, y)}(x - x^*)^2 - \frac{uc_2a_2(1 - m)x^*}{p(x, y)}(y - y^*)^2$$
$$- \frac{k\alpha(x - x^*)(y - y^*)}{(1 + ky)(1 + ky^*)} + \frac{uc_2(1 + a_2(1 - m)y^*) - c_1(1 + a_1(1 - m)^2x^*)}{p(x, y)}(x - x^*)(y - y^*).$$

Choosing u such that $uc_2(1 + a_2(1 - m)y^*) = c_1(1 + a_1(1 - m)^2x^*)$, we obtain

$$\frac{dV}{dt} = -\left[b - \frac{c_1a_1(1 - m)^2y^*}{p(x, y)}\right](x - x^*)^2 - u\left[e + \frac{c_2a_2(1 - m)x^*}{p(x, y)}\right](y - y^*)^2$$
$$- \frac{k\alpha}{(1 + ky)(1 + ky^*)}(x - x^*)(y - y^*),$$

which implies that $\frac{dV}{dt}$ is always negative definite if the following inequalities hold:

$$b > \frac{c_1a_1(1 - m)^2y^*}{p(x, y)}, \quad \left(\frac{k\alpha}{(1 + ky)(1 + ky^*)}\right)^2 < 4u\left(b - \frac{c_1a_1(1 - m)^2y^*}{p(x, y)}\right)\left(e + \frac{c_2a_2(1 - m)x^*}{p(x, y)}\right).$$

Therefore, System (2.1) and (2.2) is globally asymptotically stable around $E^* = (x^*, y^*)$ under the stated conditions. $\square$

Remark 2.2 It is to be noted that the level of fear k induced by the predator has an effect on the stability of coexisting equilibrium state. The Jacobian matrix J^* will have purely imaginary eigenvalues if $tr(J^*) = 0$ provided that $\det(J^*) > 0$. This assures the existence of a Hopf bifurcation point at $k = k_H$ around the interior equilibrium point E^*. Here, the critical value k_H for the Hopf bifurcation of the parameter k is

$$k_H = \frac{(1 + a_1(1 - m)x^* + a_2y^*)^2(\alpha - 2bx^* - 2ey^* - d) - [c_1(1 - m)y^*(1 + a_2y^*) + c_2(1 - m)x^*(1 + a_1(1 - m)x^*)]}{y^*[c_1(1 - m)y^*(1 + a_2y^*) + c_2(1 - m)x^*(1 + a_1(1 - m)x^*) + (1 + a_1(1 - m)x^* + a_2y^*)^2(2bx^* + 2ey^* + d)]}.$$

The transversality condition for the Hopf bifurcation around E^* is given by $\frac{d(tr(J^*))}{dk} = -\frac{\alpha y^*}{(1+ky^*)^2} \neq 0$ for $k = k_H$. This assures the existence of Hopf bifurcation around E^* for the Hopf bifurcation parameter k.

3 Stochastic model

Actually, population dynamics is inevitably affected by environmental white noise which is an important component in an ecosystem. To capture how environmental fluctuations affect system, stochastic perturbations need to be taken into account [34–37]. Furthermore, the population system may suffer sudden environmental shocks, e.g., earthquakes, hurricanes, epidemics, etc. However, consider only environmental white noise cannot explain such phenomena [38–40]. To explain these phenomena, the introduction of a jump process into the population dynamics provides a feasible and more realistic model. Then we obtain the following system of stochastic differential equation (SDE):

$$\begin{cases} dx(t) = x(t)\left\{\left[\frac{\alpha}{1+ky(t)} - bx(t) - \frac{c_1(1-m)y(t)}{1+a_1(1-m)x(t)+a_2y(t)}\right]dt + \sigma_1 dB_1(t)\right\} + \int_{\mathbb{Z}} \gamma_1(u)x(t^-)\tilde{N}(dt,du), \\ dy(t) = y(t)\left\{\left[-d - ey(t) + \frac{c_2(1-m)x(t)}{1+a_1(1-m)x(t)+a_2y(t)}\right]dt + \sigma_2 dB_2(t)\right\} + \int_{\mathbb{Z}} \gamma_2(u)y(t^-)\tilde{N}(dt,du), \end{cases}$$

$$(3.1)$$

with $x(0) = x_0 > 0, y(0) = y_0 > 0$.

To study the dynamics of stochastic system we first briefly go through some mathematical preliminaries.

3.1 Mathematical preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P} - null$ sets), $B_j(t)(t \geq 0)(j = 1,2)$ be a scalar standard Brownian motion defined on this probability space. $\sigma_j^2(j = 1,2)$ stands for the intensity of white noise. $x(t^-)$ and $y(t^-)$ represent the left limit of $x(t)$ and $y(t)$, respectively; N is a Poisson counting measure with compensator $\tilde{N}$ and characteristic measure λ on a measurable subset $\mathbb{Z}$ of $(0, \infty)$ with $\lambda(\mathbb{Z}) < \infty$ and $\tilde{N}(dt,du) = N(dt,du) - \lambda(du)dt$. The parameter $\gamma_j(u)$ is the effect of Lévy noise on the jth species. For biological reasons, we suppose that $1 + \gamma_i(u) > 0$, where $\gamma_i(u) > 0$ means the increasing of the species (due, e.g., to planting) and $-1 < \gamma_i(u) < 0$ means the decreasing of the species (due, e.g., to harvesting and epidemics), $u \in \mathbb{Z}, i = 1,2$. The Brownian motions and the Lévy jumps are mutually independent.

Now, we introduce the generalized Itô's formula with jumps (more details see, e.g., Reference [41]). Let $x(t) \in \mathbb{R}^n$ be a solution of the following stochastic differential equation with Lévy jumps:

$$dx(t) = F(x(t^-), t^-)dt + G(x(t^-), t^-)dB(t) + \int_{\mathbb{Z}} H(x(t^-), t^-, u)\tilde{N}(dt,du),$$

where $F : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$, $G : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ and $H : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{Z} \to \mathbb{R}^n$ are measurable functions. Given $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$, we define the operator LV by

$$LV(x(t), t) = V_t(x(t), t) + V_x(x(t), t)F(x(t), t) + \frac{1}{2}tr[G^T(x(t), t)V_{xx}(x(t), t)G(x(t), t)]$$

$$+ \int_{\mathbb{Z}} \{V(x(t) + H(x(t), t, u), t) - V(x(t), t) - V_x(x(t), t)H(x(t), t, u)\}\lambda(du),$$

where $V_t(x(t), t) = \frac{\partial V(x(t),t)}{\partial t}$, $V_x(x(t), t) = \left(\frac{\partial V(x(t),t)}{\partial x_1}, \ldots, \frac{\partial V(x(t),t)}{\partial x_n}\right)$, $V_{xx}(x(t), t) = \left(\frac{\partial^2 V(x(t),t)}{\partial x_i \partial x_j}\right)_{1\leq i,j\leq n}$.

Then the generalized Itô's formula with jumps is as follows:

$$dV(x(t), t) = LV(x(t), t)dt + V_x(x(t), t)G(x(t), t)dB(t) + \int_{\mathbb{Z}} \{V(x(t) + H(x(t), t, u), t) - V(x(t), t)\}\tilde{N}(dt,du).$$

From now on, we make the following fundamental assumptions on the jump-diffusion coefficients of model (3.1).

Assumption For any $p > 0$, there exist constants $K_j > 0$ $(j = 1, 2, 3)$ such that

$$\int_{\mathbb{Z}} \left\{ |\gamma_i(u)|^2 \vee [\ln(1 + \gamma_i(u))]^2 \right\} \lambda(du) \leq K_1 < \infty,$$

$$\int_{\mathbb{Z}} \left\{ \gamma_i(u) - \ln(1 + \gamma_i(u)) \right\} \lambda(du) \leq K_2 < \infty,$$

$$\int_{\mathbb{Z}} \left\{ (1 + \gamma_i(u))^p - 1 - p\gamma_i(u) \right\} \lambda(du) \leq K_3 < \infty, \ p \geq 1,$$

for any $i = 1, 2$. These inequalities imply that the intensity of Lévy noise cannot be too strong, otherwise, the solution of System (3.1) may explode in some finite time.

3.2   Existence and boundedness of solutions

We know, that for any given initial value a SDE will possess a unique global solution if its coefficients satisfy linear growth conditions and local Lipschitz conditions [42]. It is to be noted that the coefficients of our concerned SDE do not satisfy linear growth conditions but are locally Lipschitz continuous. So, it might be possible that the solution of the stochastic system could explode at a finite time. Now in order to show that the solution of our system is global, we will use Lyapunov analysis method.

Theorem 3.1 *Under the assumption above, for any given initial value $(x_0, y_0) \in \mathbb{R}_+^2$, System (3.1) will possess a unique solution $(x(t), y(t))$, for all $t \geqslant 0$, and the solution will remain in $\mathbb{R}_+^2$ with probability 1.*

Proof. Since the coefficients of System (3.1) are locally Lipschitz continuous, by the stochastic differential equation theory, for any given initial condition $(x_0, y_0) \in \mathbb{R}_+^2$, there is a unique local solution $(x(t), y(t))$ for $t \in [0, \tau_e)$, where τ_e is the explosion time. To show that the solution is global, we need to prove that $\tau_e = \infty$ a.s. Let $r_0 > 0$ be sufficiently large such that both x_0 and y_0 lies in the interval $[1/r_0, r_0]$. For each integer $r \geqslant r_0$, we define the stopping time by

$$\tau_r = \inf \left\{ t \in [0, \tau_e) : x(t) \notin \left(\frac{1}{r}, r \right) \text{ or } y(t) \notin \left(\frac{1}{r}, r \right) \right\}.$$

As usual, we assume the infimum of the empty set to be equal to ∞. Set $\tau_\infty = \lim_{r \to \infty} \tau_r$ and since τ_r is nondecreasing as $r \to \infty$, then $\tau_\infty \leqslant \tau_e$ a.s. So it is enough to prove that $\tau_\infty = \infty$ a.s. If not, then there exist $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leqslant T\} > \varepsilon$. Thus, by denoting $\Omega_r = \{\tau_r \leqslant T\}$, there exists $r_1 \geq r_0$ such that

$$\mathbb{P}(\Omega_r) \geqslant \varepsilon \quad \text{for all} \quad r \geqslant r_1. \tag{3.2}$$

Define a C^2-function $V_4 \colon \mathbb{R}_+^2 \to \mathbb{R}_+^2$ by $V_4(x, y) = x - 1 - \ln x + y - 1 - \ln y$ which is non-negative. If $(x(t), y(t)) \in \mathbb{R}_+^2$. By using Itô's formula, we get

$$
\begin{aligned}
dV_4(x, y) &= LV_4(x, y)dt + \sigma_1(x - 1)dB_1(t) + \sigma_2(y - 1)dB_2(t) \\
&\quad + \int_{\mathbb{Z}} \left\{ [\gamma_1(u)x - \ln(1 + \gamma_1(u))] + [\gamma_2(u)y - \ln(1 + \gamma_2(u))] \right\} \tilde{N}(dt, du),
\end{aligned}
$$

where

$$
\begin{aligned}
LV_4(x, y) &= (x - 1)\left[\frac{\alpha}{1 + ky} - bx - \frac{c_1(1 - m)y}{1 + a_1(1 - m)x + a_2 y} \right] + \left(1 - \frac{1}{y}\right) \frac{c_2(1 - m)xy}{1 + a_1(1 - m)x + a_2 y} \\
&\quad - (y - 1)(d + ey) + \frac{\sigma_1^2 + \sigma_2^2}{2} \\
&\quad + \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))] \lambda(du) + \int_{\mathbb{Z}} [\gamma_2(u) - \ln(1 + \gamma_2(u))] \lambda(du) \\
&\leqslant x\left[\frac{\alpha}{1 + ky} - bx - \frac{c_1(1 - m)y}{1 + a_1(1 - m)x + a_2 y} \right] - \left[\frac{\alpha}{1 + ky} - bx - \frac{c_1(1 - m)y}{1 + a_1(1 - m)x + a_2 y} \right]
\end{aligned}
$$

$$- y(d + ey) + (d + ey) + \frac{c_2(1-m)xy}{1 + a_1(1-m)x + a_2 y} + \frac{\sigma_1^2 + \sigma_2^2}{2}$$

$$+ \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))] \, \lambda(du) + \int_{\mathbb{Z}} [\gamma_2(u) - \ln(1 + \gamma_2(u))] \, \lambda(du)$$

$$\leqslant - bx^2 + (\alpha + b)x - ey^2 + \left(e - d + c_1(1-m) + \frac{c_2}{a_1}\right) y + d + \frac{\sigma_1^2 + \sigma_2^2}{2} + 2K_2$$

$$\leqslant \frac{(\alpha + b)^2}{4b} + \frac{\left(e - d + c_1(1-m) + \frac{c_2}{a_1}\right)^2}{4e} + d + \frac{\sigma_1^2 + \sigma_2^2}{2} + 2K_2$$

$$=: K > 0,$$

where K is a positive constant. Thus

$$dV_4(x,y) \leqslant K dt + \sigma_1(x-1)dB_1(t) + \sigma_2(y-1)dB_2(t)$$

$$+ \int_{\mathbb{Z}} [\gamma_1(u)x - \ln(1 + \gamma_1(u))] \, \tilde{N}(dt, du) + \int_{\mathbb{Z}} [\gamma_2(u)y - \ln(1 + \gamma_2(u))] \, \tilde{N}(dt, du). \qquad (3.3)$$

Integrating both sides of (3.3) from 0 to $\tau_r \wedge T$, we obtain

$$\int_0^{\tau_r \wedge T} dV_4(x(t), y(t)) \leqslant \int_0^{\tau_r \wedge T} K dt + \int_0^{\tau_r \wedge T} \sigma_1(x-1)dB_1(t) + \int_0^{\tau_r \wedge T} \sigma_2(x-1)dB_2(t)$$

$$+ \int_0^{\tau_r \wedge T} \int_{\mathbb{Z}} [\gamma_1(u)x - \ln(1 + \gamma_1(u))] \, \tilde{N}(dt, du)$$

$$+ \int_0^{\tau_r \wedge T} \int_{\mathbb{Z}} [\gamma_2(u)y - \ln(1 + \gamma_2(u))] \, \tilde{N}(dt, du).$$

Taking expectations of the above inequality leads to

$$EV_4(x_{\tau_r \wedge T}, y_{\tau_r \wedge T}) \leqslant V_4(x_0, y_0) + KE(\tau_r \wedge T) \leqslant V_4(x_0, y_0) + KT.$$

This last inequality cannot hold as $r \to \infty$ since, being either x_{τ_r} or y_{τ_r} equals either r or $1/r$, we have $V_4(x_{\tau_r \wedge T}, y_{\tau_r \wedge T}) \geqslant (r - 1 - \ln r) \wedge (1/r - 1 + \ln r)$ and see (3.2), $\mathbb{P}(\Omega_r) \geqslant \varepsilon > 0$ for all $r \geqslant r_1$. Therefore, we have $\tau_\infty = \infty$ a.s. The proof is complete. $\qquad \square$

Now we prove the boundedness of the moments of $x(t)$ and $y(t)$.

Theorem 3.2 *Let $X(t) = (x(t), y(t))$ be a solution of System (3.1). For any initial value $X_0 = (x_0, y_0) \in \mathbb{R}_+^2$, there exists $M_j(n) > 0$ $(j = 1, 2)$ such that*

$$\begin{cases} \limsup\limits_{t \to \infty} E(x^n(t)) \leqslant M_1(n), \\ \limsup\limits_{t \to \infty} E(y^n(t)) \leqslant M_2(n), \end{cases} \quad \text{for any } n \geqslant 1. \qquad (3.4)$$

Proof. Applying Itô's formula to the first equation of (3.1), we can easily obtain

$$d(e^t x^n(t)) = n e^t x^n(t) \left[\frac{1 + \tilde{\gamma}_{1n}}{n} + \frac{\alpha}{1 + ky(t)} - bx(t) - \frac{c_1(1-m)y(t)}{1 + a_1(1-m)x(t) + a_2 y(t)} + \frac{n-1}{2} \sigma_1^2 \right] dt$$

$$+ n e^t x^n(t) \sigma_1 dB_1(t) + e^t x^n(t) \int_{\mathbb{Z}} [(1 + \gamma_1(u))^n - 1] \, \tilde{N}(dt, du), \qquad (3.5)$$

where $\tilde{\gamma}_{1n} = \int_{\mathbb{Z}} \{(1 + \gamma_1(u))^n - 1 - n\gamma_1(u)\} \lambda(du)$. Integrating the two sides of (3.5) and taking expectations leads to

$$E(e^t x^n(t)) = x_0^n + n \int_0^t \left\{ E(e^s x^n(s)) \left[\frac{1 + \tilde{\gamma}_{1n}}{n} + \frac{\alpha}{1 + ky(t)} - \frac{c_1(1-m)y(t)}{1 + a_1(1-m)x(t) + a_2 y(t)} + \frac{n-1}{2} \sigma_1^2 \right] \right.$$

$$- bE(e^s x^{n+1}(s))\bigg\} ds$$

$$\leqslant x_0^n + n \int_0^t \left\{ E(e^s x^n(s)) \left[\frac{1 + \tilde{\gamma}_{1n}}{n} + \alpha + \frac{n-1}{2}\sigma_1^2 \right] - bE(e^s x^{n+1}(s)) \right\} ds$$

$$= x_0^n + nE \int_0^t h(x(s))e^s ds,$$

where $h(x) = x^n \left\{ \left[\frac{1 + \tilde{\gamma}_{1n}}{n} + \alpha + \frac{n-1}{2}\sigma_1^2 \right] - bx \right\}$. In order to find the maximum value of $h(x)$, we first calculate $h'(x)$ and obtain

$$h'(x) = nx^{n-1} \left\{ \left[\frac{1 + \tilde{\gamma}_{1n}}{n} + \alpha + \frac{n-1}{2}\sigma_1^2 \right] - bx \right\} + x^n(-b)$$

$$= x^{n-1} \left\{ n \left[\frac{1 + \tilde{\gamma}_{1n}}{n} + \alpha + \frac{n-1}{2}\sigma_1^2 \right] - b(n+1)x \right\}.$$

When $x = \frac{n\left[\frac{1+\tilde{\gamma}_{1n}}{n} + \alpha + \frac{n-1}{2}\sigma_1^2 \right]}{b(n+1)}$, we get the critical point by getting $h'(x) = 0$. Further, we notice that $h''(x) < 0$ at the critical point and maximum value at the critical point is given by

$$h_{\max} = \left(\frac{n}{b} \right)^n \left(\frac{\frac{1+\tilde{\gamma}_{1n}}{n} + \alpha + \frac{n-1}{2}\sigma_1^2}{n+1} \right)^{n+1}.$$

Therefore,

$$E(e^t x^n(t)) \leqslant x_0^n + nE \int_0^t h_{\max} e^s ds$$

$$\leqslant x_0^n + \left(\frac{n}{n+1} \right)^{n+1} \frac{\left[\frac{1+\tilde{\gamma}_{1n}}{n} + \alpha + \frac{n-1}{2}\sigma_1^2 \right]^{n+1}}{b^n}(e^t - 1),$$

i.e.,

$$E(x^n(t)) \leqslant \left\{ x_0^n - \left(\frac{n}{n+1} \right)^{n+1} \frac{\left[\frac{1+\tilde{\gamma}_{1n}}{n} + \alpha + \frac{n-1}{2}\sigma_1^2 \right]^{n+1}}{b^n} \right\} e^{-t}$$

$$+ \left(\frac{n}{n+1} \right)^{n+1} \frac{\left[\frac{1+\tilde{\gamma}_{1n}}{n} + \alpha + \frac{n-1}{2}\sigma_1^2 \right]^{n+1}}{b^n}.$$

One can observe that for $t = 0$, $E(x^n(t)) = x_0^n$. And when $t \to \infty$,

$$\limsup_{t\to\infty} E(x^n(t)) \leqslant \left(\frac{n}{n+1} \right)^{n+1} \frac{\left[\frac{1+\tilde{\gamma}_{1n}}{n} + \alpha + \frac{n-1}{2}\sigma_1^2 \right]^{n+1}}{b^n} := M_1(n).$$

Therefore, we conclude that $\limsup_{t\to\infty} E(x^n(t)) \leqslant M_1(n)$ for $n \geqslant 1$.

Similarly for predator species, we have

$$d(e^t y^n(t)) = ne^t y^n(t) \left[\frac{1 + \tilde{\gamma}_{2n}}{n} - d - ey(t) + \frac{c_2(1-m)x(t)}{1 + a_1(1-m)x(t) + a_2 y(t)} + \frac{n-1}{2}\sigma_2^2 \right] dt$$

$$+ ne^t y^n(t)\sigma_2 dB_2(t) + e^t y^n(t) \int_{\mathbb{Z}} [(1 + \gamma_2(u))^n - 1] \tilde{N}(dt, du). \tag{3.6}$$

where $\tilde{\gamma}_{2n} = \int_{\mathbb{Z}} \{(1 + \gamma_2(u))^n - 1 - n\gamma_2(u)\} \lambda(du)$. Integrating both sides of (3.6) from 0 to t and taking expectation, we have

$$E(e^t y^n(t)) = y_0^n + n \int_0^t E(e^s y^n(s)) \left[\frac{1 + \tilde{\gamma}_{2n}}{n} - d - ey(t) + \frac{c_2(1-m)x(t)}{1 + a_1(1-m)x(t) + a_2 y(t)} + \frac{n-1}{2}\sigma_2^2 \right] ds$$

$$\leqslant y_0^n + n \int_0^t E(e^s y^n(s)) \left\{ \left[\frac{1 + \tilde{\gamma}_{2n}}{n} + \frac{c_2}{a_1} + \frac{n-1}{2}\sigma_2^2 \right] - ey(t) \right\} ds$$

$$\leqslant y_0^n + nE \int_0^t e^s \left(\frac{n}{e} \right)^n \left(\frac{\frac{1+\tilde{\gamma}_{2n}}{n} + \frac{c_2}{a_1} + \frac{n-1}{2}\sigma_2^2}{n+1} \right)^{n+1} ds$$

$$\leqslant y_0^n + \left(\frac{n}{n+1} \right)^{n+1} \frac{\left[\frac{1+\tilde{\gamma}_{2n}}{n} + \frac{c_2}{a_1} + \frac{n-1}{2}\sigma_2^2 \right]^{n+1}}{e^n} (e^t - 1).$$

That is,

$$E(y^n(t)) \leqslant \left\{ y_0^n - \left(\frac{n}{n+1} \right)^{n+1} \frac{\left[\frac{1+\tilde{\gamma}_{2n}}{n} + \frac{c_2}{a_1} + \frac{n-1}{2}\sigma_2^2 \right]^{n+1}}{e^n} \right\} e^{-t}$$

$$+ \left(\frac{n}{n+1} \right)^{n+1} \frac{\left[\frac{1+\tilde{\gamma}_{2n}}{n} + \frac{c_2}{a_1} + \frac{n-1}{2}\sigma_2^2 \right]^{n+1}}{e^n}.$$

By the similar reason as above, we have

$$\limsup_{t \to \infty} E(y^n(t)) \leqslant \left(\frac{n}{n+1} \right)^{n+1} \frac{\left[\frac{1+\tilde{\gamma}_{2n}}{n} + \frac{c_2}{a_1} + \frac{n-1}{2}\sigma_2^2 \right]^{n+1}}{e^n} := M_2(n), n \geqslant 1.$$

The proof is now complete.   □

3.3   Stochastic extinction scenario

In this subsection, we investigate the extinction criterion of our stochastic System (3.1). Before going to the main results, we want to highlight the following definition [43].

Definition 3.1 Let θ be a positive constant. Species p is said to be extinct exponentially if there exists $\limsup_{t\to\infty} t^{-1} \ln p(t) < -\theta$ a.s. This fact implies that species p is also extinct with $\lim_{t\to\infty} p(t) = 0$ a.s.

To determine the conditions of extinction of a species in an ecosystem is very important from biological point of view. In this regard, we have the following theorem:

Theorem 3.3 *Let* $(x(t), y(t))$ *be the solution of the stochastic System* (3.1) *initiating from any value* $(x_0, y_0) \in \mathbb{R}_+^2$. *Then both the species extinct exponentially with probability one if* $\alpha < \frac{\sigma_1^2}{2}$ *and* $\frac{c_2}{a_1} < d + \frac{\sigma_2^2}{2}$.

Proof. Applying Itô's formula to the equations of (3.1) we already obtained

$$d(\ln x(t)) = \left\{ \frac{\alpha}{1 + ky(t)} - bx(t) - \frac{c_1(1-m)y(t)}{1 + a_1(1-m)x(t) + a_2 y(t)} - \frac{\sigma_1^2}{2} - \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))] \lambda(du) \right\} dt$$
$$+ \sigma_1 dB_1(t) + \int_{\mathbb{Z}} \ln(1 + \gamma_1(u)) \tilde{N}(dt, du), \qquad (3.7)$$

$$d(\ln y(t)) = \left\{ -d - ey(t) + \frac{c_2(1-m)x(t)}{1 + a_1(1-m)x(t) + a_2 y(t)} - \frac{\sigma_2^2}{2} - \int_{\mathbb{Z}} [\gamma_2(u) - \ln(1 + \gamma_2(u))] \lambda(du) \right\} dt$$
$$+ \sigma_2 dB_2(t) + \int_{\mathbb{Z}} \ln(1 + \gamma_2(u)) \tilde{N}(dt, du). \qquad (3.8)$$

Now, from (3.7)

$$d(\ln x(t)) \leqslant \left[\alpha - bx(t) - \frac{\sigma_1^2}{2} \right] dt + \sigma_1 dB_1(t) + \int_{\mathbb{Z}} \ln(1 + \gamma_1(u)) \tilde{N}(dt, du). \qquad (3.9)$$

Now, we take $S(x) = \alpha - bx - \frac{\sigma_1^2}{2}$, here we intend to get the supremum value of $S(x)$, we obtain $S'(x) = -b < 0$. So $S(x)$ is a decreasing function of x on $[0, \infty)$ and hence supremum value of $S(x)$ is

$$\sup_{t \geq 0} S(x(t)) = S(0) = \alpha - \frac{\sigma_1^2}{2}.$$

Integrating both sides of (3.9) from 0 to t leads to

$$\ln x(t) - \ln x_0 \leqslant \int_0^t \left(\alpha - \frac{\sigma_1^2}{2} \right) ds + P_1(t) \leqslant t \left(\alpha - \frac{\sigma_1^2}{2} \right) + P_1(t). \tag{3.10}$$

where $P_1(t) = \int_0^t \sigma_1 dB_1(s) + \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_1(u)) \tilde{N}(ds, du)$. By strong law of large numbers for local martingales, we obtain

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \sigma_1 dB_1(s) = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_1(u)) \tilde{N}(ds, du) = 0 \quad a.s.,$$

i.e.,

$$\lim_{t \to \infty} \frac{P_1(t)}{t} = 0 \quad a.s. \tag{3.11}$$

Thus

$$\limsup_{t \to \infty} \frac{\ln x(t)}{t} \leqslant \limsup_{t \to \infty} \left[\frac{\ln x_0}{t} + \frac{t \left(\alpha - \frac{\sigma_1^2}{2} \right)}{t} + \frac{P_1(t)}{t} \right] = \alpha - \frac{\sigma_1^2}{2}.$$

Now, using the condition $\alpha < \frac{\sigma_1^2}{2}$, we obtain

$$\limsup_{t \to \infty} \frac{\ln x(t)}{t} \leqslant \alpha - \frac{\sigma_1^2}{2} < 0.$$

This fact implies that the prey species is extinct exponentially.

Now, in a similar way we can show that if the condition $\frac{c_2}{a_1} < d + \frac{\sigma_2^2}{2}$ holds then the predator species is extinct exponentially. $\qquad\square$

3.4 Stochastic persistence

In the present section our aim is to find out sufficient conditions for the persistence of both the species under certain parametric restrictions.

Definition 3.2 ([44]) A population $z(t)$ is said to be persistent or persistent in mean if $\liminf_{t \to \infty} \langle z(t) \rangle > 0$, where $\langle z(t) \rangle = \frac{1}{t} \int_0^t z(r) dr$, a.s. and the stochastic System (3.1) is said to be persistent if both the populations $x(t)$ and $y(t)$ are all persistent.

Lemma 3.1 *Consider the following one-dimensional stochastic system:*

$$dz(t) = z(t) \left(\frac{\alpha}{1 + ky(t)} - bz(t) \right) dt + \sigma_1 z(t) dB_1(t) + \int_{\mathbb{Z}} \gamma_1(u) z(t^-) \tilde{N}(dt, du), \quad \text{with } z(0) = x_0. \tag{3.12}$$

i. *If $\alpha - \frac{\sigma_1^2}{2} < 0$, then $\lim_{t \to \infty} z(t) = 0$ a.s.*

ii. *If $\alpha - \frac{\sigma_1^2}{2} > 0$, then $\lim_{t \to \infty} \frac{1}{t} \int_0^t z(s) ds = \alpha - \frac{\sigma_1^2}{2}$.*

iii. *For $n \geqslant 1$, $\limsup_{t \to \infty} E(z^n(t)) \leqslant M_1(n)$, where $M_1(n) = \left(\frac{n}{n+1} \right)^{n+1} \frac{\left[\frac{1 + \tilde{\gamma}_{1n}}{n} + \alpha + \frac{n-1}{2} \sigma_1^2 \right]^{n+1}}{b^n}$.*

Proof. The above conclusion can be obtained by the similar calculation as Theorem 3.2 and Theorem 3.3. So we omit it here.   □

Lemma 3.2 (See Lemma 5.1 in [2]) *Suppose* $Z(t) \in \mathbb{C}(\Omega \times [0, \infty), \mathbb{R}_+)$.

     i. *If* T, δ *and* δ_0 *are positive constants such that*

$$\ln Z(t) \geqslant \delta t - \delta_0 \int_0^t Z(s)ds + \sum_{i=1}^n \alpha_i B_i(t) \quad a.s. \ \forall t \geqslant T,$$

     where α_i *are constants for* $i = 1, 2, \ldots, n$, *then*

$$\liminf_{t \to \infty} \langle Z(t) \rangle \geqslant \frac{\delta}{\delta_0} \quad a.s.$$

     ii. *If* T, δ *and* δ_0 *are positive constants such that*

$$\ln Z(t) \leqslant \delta t - \delta_0 \int_0^t Z(s)ds + \sum_{i=1}^n \alpha_i B_i(t) \quad a.s. \ \forall t \geqslant T,$$

     where α_i *are constants for* $i = 1, 2, \ldots, n$, *then*

$$\begin{cases} \limsup_{t \to \infty} \langle Z(t) \rangle \leqslant \frac{\delta}{\delta_0} & a.s. \quad if \ \delta > 0, \\ \lim_{t \to \infty} \langle Z(t) \rangle = 0, & a.s. \quad if \ \delta < 0. \end{cases}$$

Theorem 3.4 *The prey population* $x(t)$ *of System* (3.1) *will be persistent in mean if* $\frac{c_1(1-m)}{a_2} + \frac{\sigma_1^2}{2} + K_2 < \frac{\alpha}{1+kM_2(1)}$. *And if* $x(t)$ *is persistent then the predator population* $y(t)$ *is also persistent if* $d + \frac{\sigma_1^2}{2} + K_2 + \frac{c_2}{a_1^2(1-m)X^*} < \frac{c_2}{a_1}$. *Moreover if both conditions hold true then both species are persistent.*

Proof. Using stochastic integration process of (3.7) and dividing both sides by t, we get

$$\frac{\ln x(t) - \ln x_0}{t} = \frac{1}{t} \int_0^t \frac{\alpha}{1+ky(r)}dr - \frac{\sigma_1^2}{2} - \frac{b}{t} \int_0^t x(r)dr - \frac{c_1(1-m)}{t} \int_0^t \frac{y(r)}{1+a_1(1-m)x(r)+a_2 y(r)}dr$$
$$- \frac{1}{t} \int_0^t \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1+\gamma_1(u))] \lambda(du)dr - \frac{P_1(t)}{t}, \tag{3.13}$$

where $P_1(t) = \int_0^t \sigma_1 dB_1(r) + \int_0^t \int_{\mathbb{Z}} \ln(1+\gamma_1(u))\tilde{N}(dr, du)$.
Let $z(t)$ be a solution to (3.12), then consider $\ln z(t)$ and apply Itô's formula, so we have

$$d(\ln z(t)) = \left\{ \frac{\alpha}{1+ky(t)} - bz(t) - \frac{\sigma_1^2}{2} - \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1+\gamma_1(u))] \right\} dt + \sigma_1 dB_1(t) + \int_{\mathbb{Z}} \ln(1+\gamma_1(u))\tilde{N}(dt, du).$$

Again using stochastic integration process of above equation and dividing both sides by t, we get

$$\frac{\ln z(t) - \ln x_0}{t} = \frac{1}{t} \int_0^t \frac{\alpha}{1+ky(r)}dr - \frac{\sigma_1^2}{2} - \frac{b}{t} \int_0^t z(r)dr - \frac{1}{t} \int_0^t \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1+\gamma_1(u))] \lambda(du)dr - \frac{P_1(t)}{t}. \tag{3.14}$$

From (3.13) and (3.14), then we have

$$0 \geqslant \frac{\ln x(t) - \ln z(t)}{t} = \frac{b}{t} \int_0^t (z(r) - x(r))\, dr - \frac{c_1(1-m)}{t} \int_0^t \frac{y(r)}{1+a_1(1-m)x(r)+a_2 y(r)}dr,$$

which implies $x(t) \leqslant z(t)$. Now, using by Theorem 3.2, we obtain from (3.7)

$$d(\ln x(t)) \geqslant \left(\frac{\alpha}{1+kM_2(1)} - \frac{c_1(1-m)}{a_2} - \frac{\sigma_1^2}{2} - K_2 \right) dt - bx(t)dt + \sigma_1 dB_1(t) + \int_{\mathbb{Z}} \ln(1+\gamma_1(u))\tilde{N}(dt, du).$$

Now, integrating both sides of the above equation from 0 to t and then dividing by t, we get

$$\frac{\ln x(t) - \ln x_0}{t} \geqslant \left(\frac{\alpha}{1 + kM_2(1)} - \frac{c_1(1-m)}{a_2} - \frac{\sigma_1^2}{2} - K_2\right) - \frac{b}{t}\int_0^t x(r)dr + \frac{P_1(t)}{t}. \qquad (3.15)$$

Applying Lemma 3.2-item i, we finally obtain

$$\liminf_{t\to\infty}\langle x(t)\rangle \geqslant \frac{\frac{\alpha}{1+kM_2(1)} - \frac{c_1(1-m)}{a_2} - \frac{\sigma_1^2}{2} - K_2}{b} > 0 \ a.s., \ \text{since} \ \frac{c_1(1-m)}{a_2} + \frac{\sigma_1^2}{2} + K_2 < \frac{\alpha}{1 + kM_2(1)}.$$

Hence, the prey species persists in the system. Now, for the persistence of predator species, the persistence of prey species is necessary for obvious reasons. We proved earlier that prey species persists in the system for certain condition. Thus, we assume that there exists a minimum, $X^* > 0$. Now, from (3.8) we get

$$d(\ln y(t)) \geqslant \left(\frac{c_2}{a_1} - d - \frac{\sigma_1^2}{2} - K_2 - \frac{c_2}{a_1^2(1-m)X^*}\right)dt - \left(\frac{c_2a_2}{a_1^2(1-m)X^*} + e\right)ydt$$
$$+ \sigma_2 dB_2(t) + \int_{\mathbb{Z}} \ln(1 + \gamma_2(u))\tilde{N}(dt, du).$$

Now, applying stochastic integrating process and then dividing by t we obtain

$$\frac{\ln y(t) - \ln y_0}{t} \geqslant \left(\frac{c_2}{a_1} - d - \frac{\sigma_1^2}{2} - K_2 - \frac{c_2}{a_1^2(1-m)X^*}\right) - \left(\frac{c_2a_2}{a_1^2(1-m)X^*} + e\right)\int_0^t y(r)dr + \frac{P_2(t)}{t}. \quad (3.16)$$

where $P_2(t) = \int_0^t \sigma_2 dB_2(r) + \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_2(u))\tilde{N}(dr, du)$ and $\lim_{t\to\infty}\frac{P_2(t)}{t} = 0$ a.s. Finally, applying result i. of Lemma 3.2, we obtain

$$\liminf_{t\to\infty}\langle y(t)\rangle \geqslant \frac{\frac{c_2}{a_1} - d - \frac{\sigma_1^2}{2} - K_2 - \frac{c_2}{a_1^2(1-m)X^*}}{\frac{c_2a_2}{a_1^2(1-m)X^*} + e} > 0 \ a.s., \ \text{when} \ d + \frac{\sigma_1^2}{2} + K_2 + \frac{c_2}{a_1^2(1-m)X^*} < \frac{c_2}{a_1}.$$

Therefore, if the conditions $\frac{c_1(1-m)}{a_2} + \frac{\sigma_1^2}{2} + K_2 < \frac{\alpha}{1+kM_2(1)}$ and $d + \frac{\sigma_1^2}{2} + K_2 + \frac{c_2}{a_1^2(1-m)X^*} < \frac{c_2}{a_1}$ hold simultaneously, then the predator species persists in the system. It is to be noted that under the above conditions the stochastic System (3.1) is persistent. $\qquad \square$

4 Numerical simulations

In this section, we will perform some numerical simulations using MATLAB R2022a to illustrate the effect of white noise, Lévy noise, fear effect and a prey refuge on the dynamics of the deterministic System (2.1) (2.2) and stochastic system (3.1). In this section, we always take the following parameter values:

$$\alpha = 2, \ b = 0.01, \ d = 0.02, \ e = 0.01, \ c_1 = 0.6, \ c_2 = 0.3, \ a_1 = 1.3, \ a_2 = 0.9, \ k = 0.65, \ m = 0.01, \qquad (4.1)$$

and $\mathbb{Z} = (0, +\infty)$, $\lambda(\mathbb{Z}) = 1$ with initial value $(x_0, y_0) = (0.6, 0.6)$.

For the above set of parameter values, the deterministic system has three non-negative equilibrium points, namely trivial equilibrium point $E^0(0,0)$, predator free equilibrium point $E^1(200, 0)$ and the co-existing equilibrium point $E^*(3.003, 6.399)$. Stability of positive steady state has been shown in Figure 1 through time series and phase portrait of the solutions of the deterministic System (2.1) - (2.2).

To elucidate the impact of fear in the dynamics of System (2.1), we keep changing the parameter value of k (i.e., level of induced fear) while keeping all other parameter values fixed. We notice that when the parameter value of k crosses a certain threshold value, then the system becomes unstable and both the species coexist in periodic mode. Here, we observed from numeric simulations that the parameter k plays the role as Hopf bifurcation parameter which already has been established analytically in Remark 2.2. For $k = 0.8$, the eigenvalues cross the imaginary axis and real part of both the eigenvalues becomes positive which leads to Hopf bifurcating periodic solution. Figure 2 exhibit the fact. Here, it is to be noted that the

size of both the populations reduce due to induced fear, although the predator species still dominates the prey species. From the Hopf bifurcation diagrams Figure 3 we notice that the stability of the model system changes when the bifurcation parameter k crosses the threshold value 0.7224, which is consistent with the value of k_H obtained with the formula in Remark 2.2

For numerical simulation of the stochastic System (3.1), we approximate the solution by Milsteins high order method [45] in Matlab. We solve System (3.1) taking the values of the parameters as that in (4.1) with low intensity of noise as $\sigma_1 = \sigma_2 = 0.1$, $\gamma_1 = \gamma_2 = 0.15$, we can compute that the conditions of Theorem 3.4 hold here. Hence System (3.1) will be persistent in the mean. Figure 4 depicts the fact. Next, we consider the case where the intensity of white noise is too large. We take $\sigma_1 = 2$, $\sigma_2 = 1$ and keep the other parameters consistent with (4.1). Then we get $\alpha \leqslant \frac{\sigma_1^2}{2}$ and $\frac{c_2}{a_1} \leqslant d + \frac{\sigma_2^2}{2}$, which means that the conditions of extinction for both species of Theorem 3.3 are satisfied. We exhibit the fact by Figure 5.

Now we consider some other cases. In order to obtain deep insights of the influences of Lévy noise, we keep the model parameter values the same but chose different values of γ_1 and γ_2, say $\gamma_1 = \gamma_2 = 0$, $\gamma_1 = \gamma_2 = 0.3$ and $\gamma_1 = \gamma_2 = 0.5$. We can obtain that System (3.1) is persistent in the mean. We can find that the Lévy jumps promote the survival of both prey and predator populations to a certain extent. In the absence of Lévy noise, population numbers of the two species remain at a low level with little difference and the volatility is smaller (see Figure 6). Besides, we numerically simulate the impact of a prey refuge to model (3.1) and choose different values of m, say $m = 0$, $m = 0.5$ and $m = 0.8$, shown in Figure 7. From Figure 7, it can be seen that as the prey refuge stength increases, the prey population density increases and the predator population density decreases. The prey refuge strength affects the prey more than the predators populations.
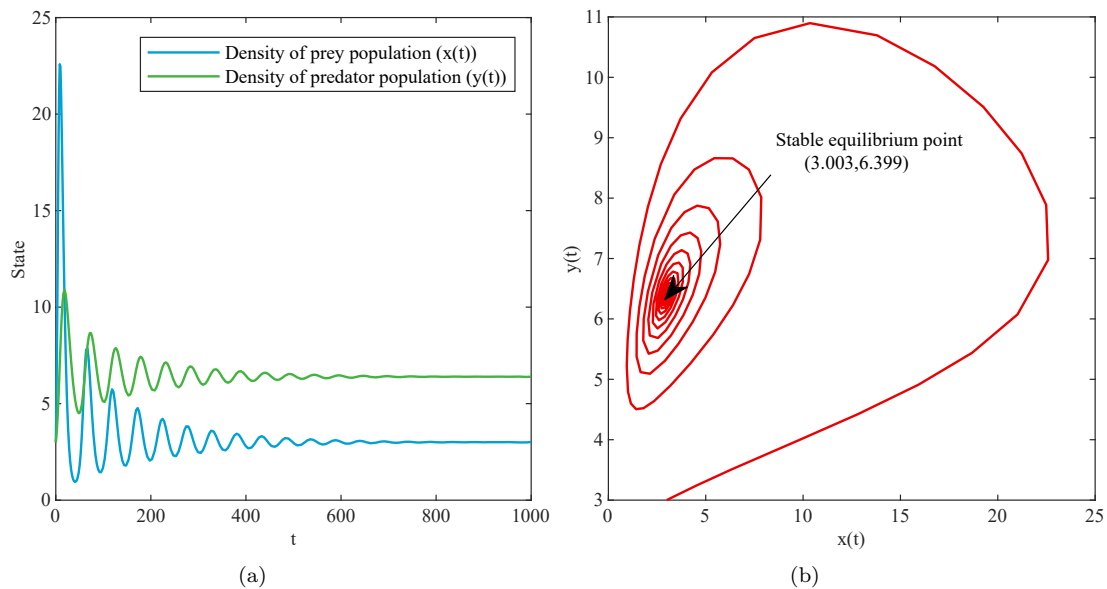

Figure 1: Time evolution and phase portrait of the prey predator species of System (2.1) for $k = 0.65$. The system exhibits stable behaviour of both prey and predator species.

5   Conclusion

In this work, we have considered and analyzed a density dependent (for predator) nonlinear ordinary differential equation model (2.1) and (2.2) with Beddington-DeAngelis functional response incorporating prey refuge and fear effect and its stochastic version (3.1). Incorporating refuge and fear effect into system provides a more realistic model. We have dealt with all the dynamics in a systematic manner. The criteria for the global stability of the coexistence equilibrium point for deterministic model have been derived using Lyapunov-like functions. We have derived the boundedness and persistence conditions for both the deterministic and stochastic models.
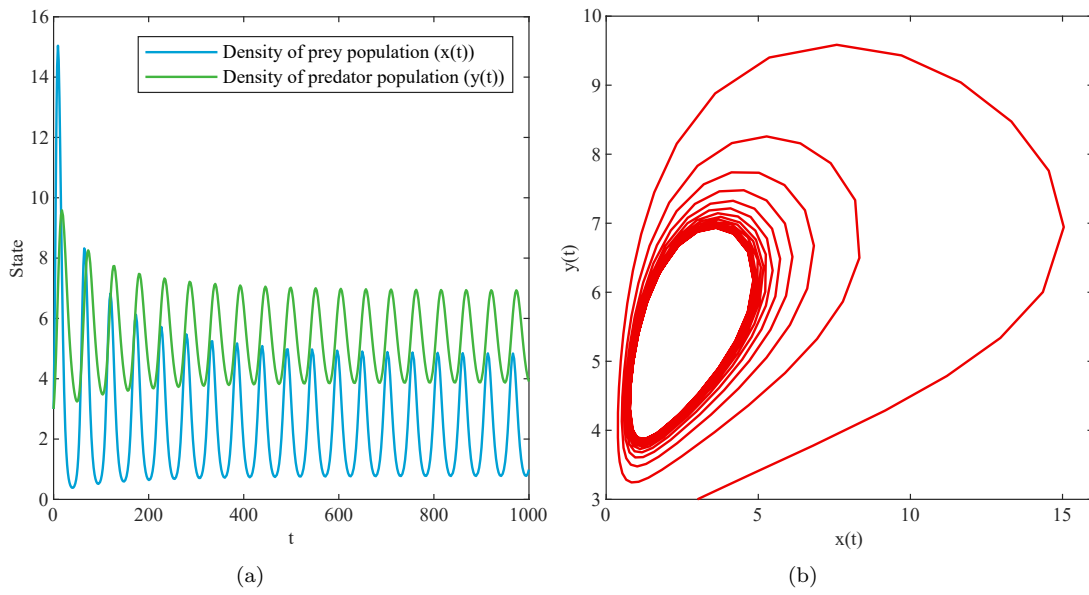
Figure 2: Time series solution and phase portrait of System (2.1) for $k = 0.8$. The system exhibits periodic coexistence of the prey and predator species in the form of a limit cycle.
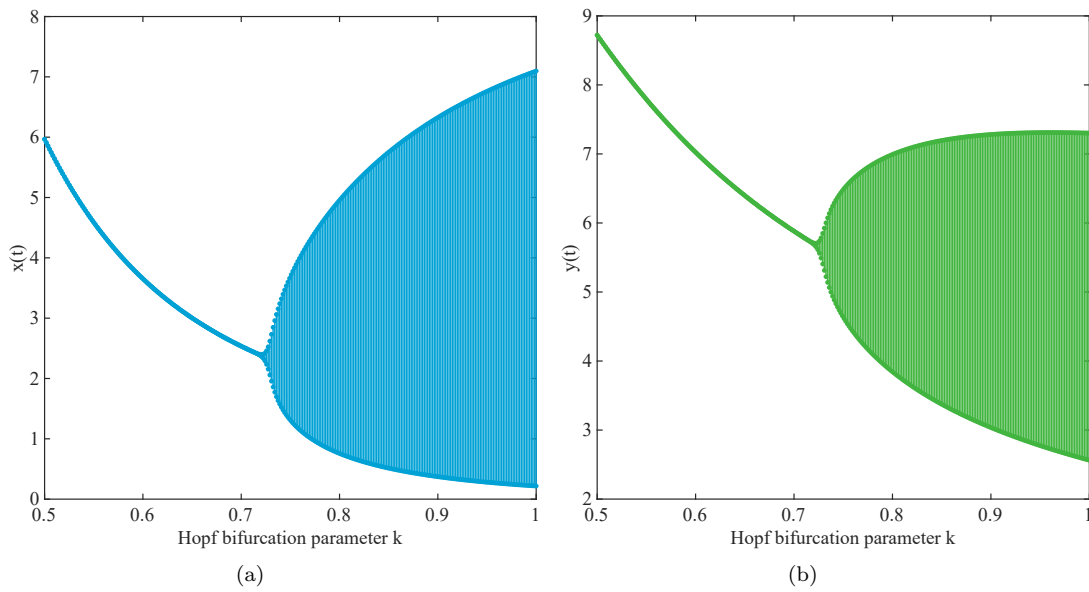

Figure 3: Hopf Bifurcation diagrams for the bifurcation parameter k show that the system becomes unstable from stable after crossing the threshold value $k = 0.7224$.
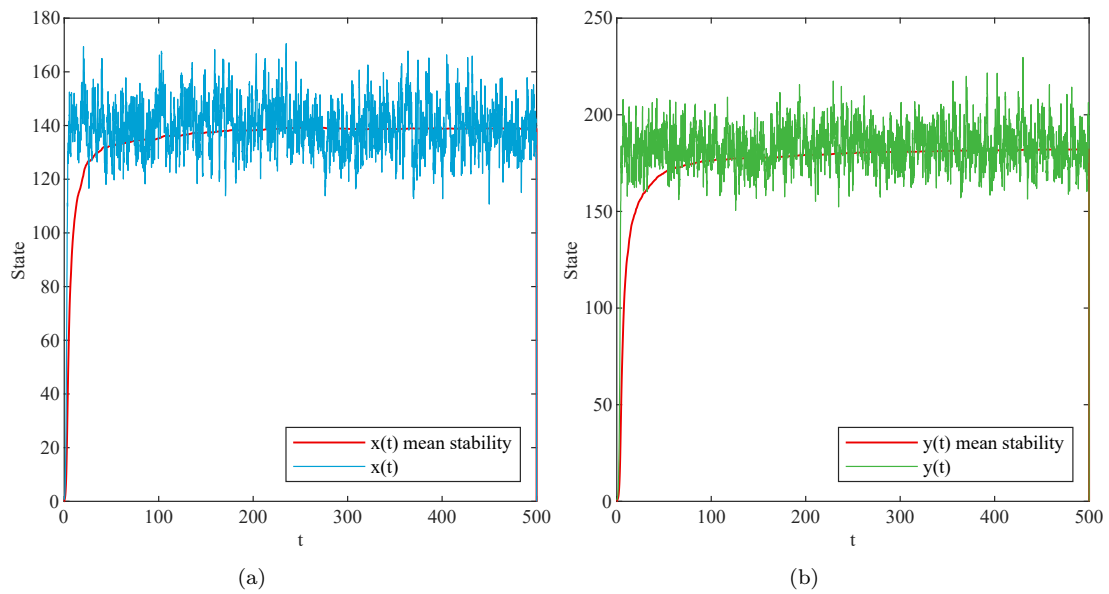
Figure 4: Trajectory images of population change in System (3.1). (a) and (b) respectively represent the persistence in the mean of prey and predator populations.
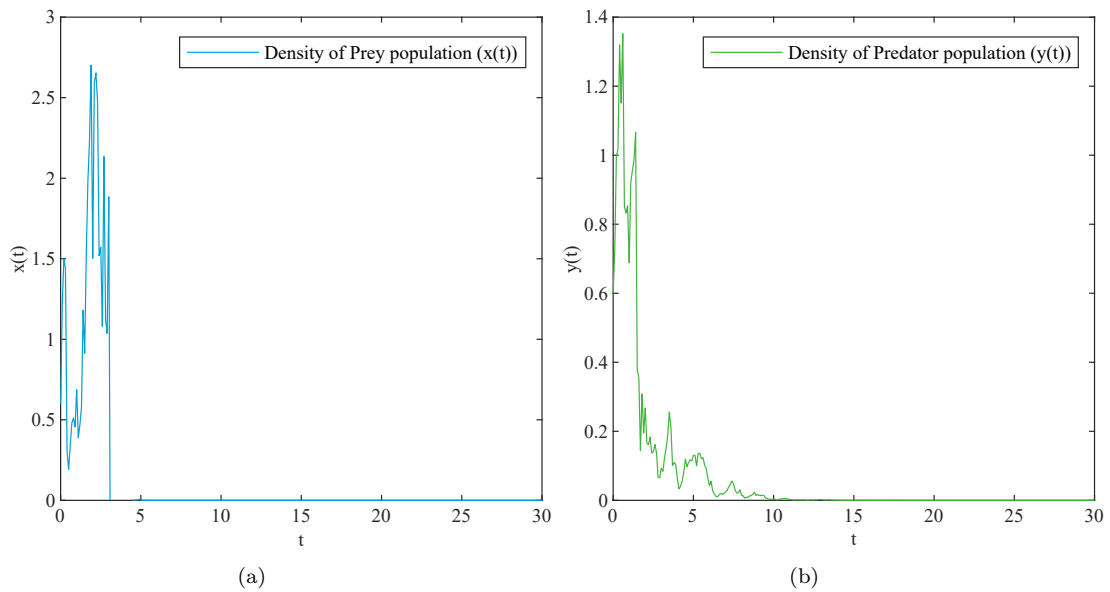

Figure 5: Time evolution of the stochastic predator-prey System (3.1) when the intensity of noise is high. These figures clearly show that high intensity of noise leads to the extinction of both species.
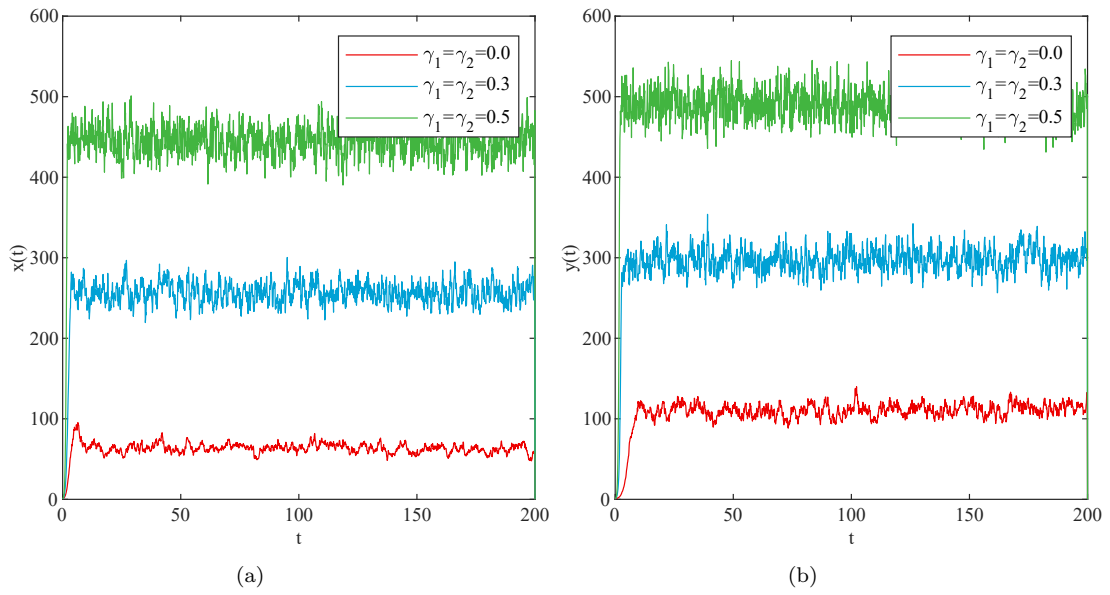
Figure 6: Population state diagrams for different intensity of Lévy noise. (a) and (b) represent time evolution of the prey and predator populations respectively, we can find that Lévy noise promotes the survival of species.
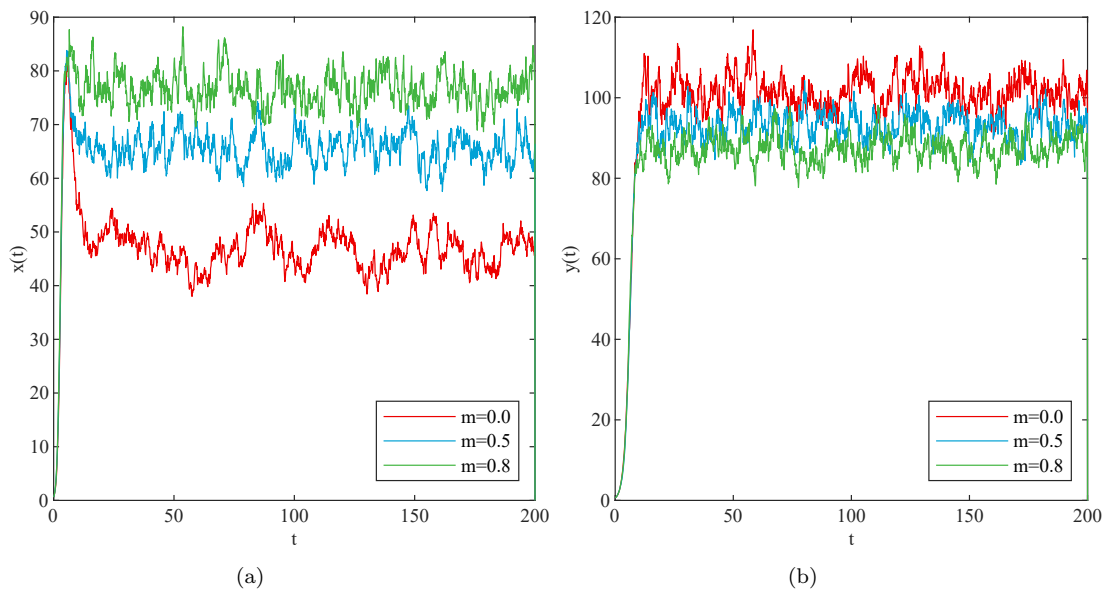

Figure 7: Population state diagrams for different values of prey refuge strength m. Numerical simulation for model (3.1) with initial value $(x_0, y_0) = (0.6, 0.6)$ and different strengths $m = 0$, $m = 0.5$ and $m = 0.8$, respectively. The other parameters are taken as in (4.1).

In order to capture the oscillatory coexistence, we performed the Hopf bifurcation analysis with level of fear as bifurcation parameter. Our analysis reveals that the periodic coexistence is possible under certain conditions. We have shown in Figure 3 that System (2.1) - (2.2) experiences the Hopf bifurcation as the fear parameter k crosses an explicit critical values.

Furthermore, population dynamics is inevitably affected by environmental white noise and sudden environmental shocks both of which are important components of ecology. To explain these phenomena, the introduction of white noise and jump process into population dynamics provides a feasible and more realistic model. We have shown that the solutions of the stochastic system will not explode within a reasonable finite time and the system will possess a unique global solution starting from any interior of the positive quadrant. We have derived criterion for stochastic extinction and persistence of both the species.

In practice, multi-species system (for example, food chain system [46]) often exhibits more complex dynamical behaviors. For such system, we believe that there may be some similar results, which is interesting and left to our future work.

Competing interests

The authors declare that they have no competing interests in this paper.

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